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## UNIQUENESS THEOREMS FOR CAUCHY INTEGRALS

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*Abstract*

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If  $\mu$  is a finite complex measure in the complex plane  $\mathbb{C}$  we denote by  $C^\mu$  its Cauchy integral defined in the sense of principal value. The measure  $\mu$  is called *reflectionless* if it is continuous (has no atoms) and  $C^\mu = 0$  at  $\mu$ -almost every point. We show that if  $\mu$  is reflectionless and its Cauchy maximal function  $C_*^\mu$  is summable with respect to  $|\mu|$  then  $\mu$  is trivial. An example of a reflectionless measure whose maximal function belongs to the “weak”  $L^1$  is also constructed, proving that the above result is sharp in its scale. We also give a partial geometric description of the set of reflectionless measures on the line and discuss connections of our results with the notion of sets of finite perimeter in the sense of De Giorgi.

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### 1. Introduction

This article discusses uniqueness theorems for Cauchy integrals of complex measures in the plane. We consider the space  $M = M(\mathbb{C})$  of finite complex measures  $\mu$  in  $\mathbb{C}$ . The Cauchy integral of a measure from  $M$  is defined in the sense of principal value. First, for any  $\mu \in M$ ,  $\varepsilon > 0$  and any  $z \in \mathbb{C}$  consider

$$C_\varepsilon^\mu(z) := \int_{|\zeta - z| > \varepsilon} \frac{d\mu(\zeta)}{\zeta - z}.$$

Consequently, the Cauchy integral of  $\mu$  can be defined as

$$C^\mu(z) := \lim_{\varepsilon \rightarrow 0} C_\varepsilon^\mu(z),$$

if the limit exists.

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Unlike the Cauchy transform on the line,  $C^\mu$  can vanish on a set of positive Lebesgue measure: consider for example  $\mu = dz$  on a closed curve, whose Cauchy transform is zero at all points outside the curve. It is natural to ask if  $C^\mu$  can also vanish on large sets with respect to  $\mu$ . If  $\mu = \delta_z$  is a single point mass, its Cauchy transform will be zero  $\mu$ -a.e. due to the above definition of  $C^\mu$  in the sense of principal value. Examples of infinite discrete measures with vanishing Cauchy transforms can also be constructed with little effort.

After that one arrives at the following corrected version of the question: Is it true that any continuous  $\mu \in M$ , such that  $C^\mu(z) = 0$  at  $\mu$ -a.e. point, is trivial? As usual, we call a measure continuous if it has no point masses. We denote the space of all finite complex continuous measures by  $M_c(\mathbb{C})$ .

This problem can also be interpreted in terms of uniqueness. Namely, if  $f$  and  $g$  are two functions from  $L^1(|\mu|)$  such that  $C^{(f-g)\mu} = 0$ ,  $\mu$ -a.e., does it imply that  $f = g$ ,  $\mu$ -a.e.? In this way it becomes a problem of injectivity of the planar Cauchy transform.

The first significant progress towards the solution of this problem was achieved by X. Tolsa and J. Verdera in [15]. It was established that the answer is positive in two important particular cases: when  $\mu$  is absolutely continuous with respect to Lebesgue measure  $m_2$  in  $\mathbb{C}$  and when  $\mu$  is a measure of linear growth with finite Menger curvature. The latter class of measures is one of the main objects in the study of the planar Cauchy transform, see for instance [11], [12] or [14].

As to the complete solution of the problem, it seemed for a while that the answer might be positive for any  $\mu \in M_c$ , see for example [15]. However, in Section 5 of the present paper we show that there exists a large set of continuous measures  $\mu$  satisfying  $C^\mu(z) = 0$ ,  $\mu$ -a.e. Following [2], we call such measures *reflectionless*. This class seems to be an intriguing new object in the theory.

On the positive side, we prove that if the maximal function associated with the Cauchy transform is summable with respect to  $|\mu|$  then  $\mu$  cannot be reflectionless, see Theorem 2.1. This result is sharp in its scale because the simplest examples of reflectionless measures produce maximal functions that lie in the “weak”  $L^1(|\mu|)$ . We prove this result in Section 2. In view of this fact, we believe that the class of continuous measures with summable Cauchy maximal functions also deserves attention.

A full description of this class and the (disjoint) class of reflectionless measures remains an **open problem**. Results of [8] imply that

reflectionless measures cannot have Hausdorff dimension less than one. Examples given in the present paper produce measures of dimension one. Whether reflectionless measures in the plane may have dimension greater than one, remains to be seen.

Let us mention that if  $\mu$  is a measure with linear growth and finite Menger curvature then its Cauchy maximal function belongs to  $L^2(|\mu|)$ , see [12], [14], and therefore is summable. This fact relates Theorem 2.1 to the aforementioned result from [15]. The latter can also be deduced in a different way, see Section 2.

From the point of view of uniqueness, our results imply that any bounded planar Cauchy transform (see Section 2 for the definition) is injective, see Corollary 2.5. This property is a clear analogue of the uniqueness results for the Cauchy integral on the line or the unit circle.

In Section 3 we discuss other applications of Theorem 2.2. They involve structural theorems of De Giorgi and his notion of a set of finite perimeter, see [4].

In Section 4 we study asymptotic behavior of the Cauchy transform near its zero set. The results of this section imply that the Radon derivative of  $\mu$  with respect to Lebesgue measure  $m_2$  vanishes a.e. on the set  $\{C^\mu = 0\}$ . In particular the set  $\{C^\mu = 0\}$  must be a zero set with respect to the variation of the absolutely continuous part of  $\mu$  which is a slight generalization of the first result of [15]. It is interesting to note that the most direct analogue of this corollary on the real line is false: it is easy to construct an absolutely continuous (with respect to  $m_1 = dx$ ) measure  $\mu \in M(\mathbb{R})$  such that  $|\mu|(\{C^\mu = 0\}) > 0$ .

Finally, in Section 5 we attempt a geometric description of the set of reflectionless measures. We give a partial description of reflectionless measures on the line in terms of so-called comb-like domains. We also provide tools for the construction of various examples of such measures. In particular, we show that the harmonic measure on any compact subset (of positive Lebesgue measure) of  $\mathbb{R}$  is reflectionless.

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## 2. Measures with summable maximal functions

If  $\mu \in M$  we denote by  $C_*^\mu(z)$  its Cauchy maximal function

$$C_*^\mu(z) := \sup_{\varepsilon > 0} |C_\varepsilon^\mu(z)|.$$

Our first result is the following uniqueness theorem.

**Theorem 2.1.** *Let  $\mu \in M_c$ . Assume that  $C_*^\mu(z) \in L^1(|\mu|)$  and that  $C^\mu(z)$  exists and vanishes  $\mu$ -a.e. Then  $\mu \equiv 0$ .*

We first prove

**Theorem 2.2.** *If  $C_*^\mu \in L^1(|\mu|)$  and  $C^\mu(z)$  exists  $\mu$ -a.e. then*

$$(1) \quad 2C^{C^\mu d\mu}(z) = 2 \int \frac{C^\mu(t) d\mu(t)}{t-z} = [C^\mu(z)]^2 \text{ for } m_2\text{-a.e. point } z \in \mathbb{C}.$$

*Proof:* Put

$$F := \left\{ z \in \mathbb{C} : \int \frac{d|\mu|(t)}{|t-z|} < \infty \right\}.$$

As  $|\mu|$  is a finite measure,

$$(2) \quad m_2(\mathbb{C} \setminus F) = 0.$$

Let  $z \in F$ . Then the integral

$$I(z, \varepsilon) = I := \iint_{|t-\zeta| > \varepsilon} d\mu(t) d\mu(\zeta) \frac{1}{t-z} \cdot \frac{1}{\zeta-z}$$

is absolutely convergent for any  $\varepsilon > 0$ .

Using the identity

$$\frac{1}{(t-z)(z-\zeta)} + \frac{1}{(z-\zeta)(\zeta-t)} + \frac{1}{(\zeta-t)(t-z)} \equiv 0$$

we obtain

$$\begin{aligned} I &= \iint_{|t-\zeta| > \varepsilon} \left[ \frac{1}{z-\zeta} \cdot \frac{1}{\zeta-t} + \frac{1}{\zeta-t} \cdot \frac{1}{t-z} \right] d\mu(t) d\mu(\zeta) \\ &= \int \frac{d\mu(\zeta)}{\zeta-z} \int_{|t-\zeta| > \varepsilon} \frac{d\mu(t)}{t-\zeta} + \int \frac{d\mu(t)}{t-z} \int_{|\zeta-t| > \varepsilon} \frac{d\mu(\zeta)}{\zeta-t} \\ &= \int C_\varepsilon^\mu(t) \frac{1}{t-z} d\mu(t) + \int C_\varepsilon^\mu(\zeta) \frac{1}{\zeta-z} d\mu(\zeta) = 2 \int \frac{C_\varepsilon^\mu(t) d\mu(t)}{t-z}. \end{aligned}$$

Put

$$E := \left\{ z \in \mathbb{C} : \int \frac{C_*^\mu(t) d|\mu|(t)}{|t-z|} < \infty \right\}.$$

By assumption, the numerator  $C_*^\mu(t) d|\mu|(t)$  is a finite measure. Therefore

$$(3) \quad m_2(\mathbb{C} \setminus E) = 0.$$

If  $z \in E$  then

$$(4) \quad \lim_{\varepsilon \rightarrow 0} \int \frac{C_{\varepsilon}^{\mu}(t) d\mu(t)}{t-z} = \int \frac{C^{\mu}(t) d\mu(t)}{t-z}.$$

This formula is true as long as  $C_{*}^{\mu} \in L^1(|\mu|)$  and the principal value  $C^{\mu}$  exists  $\mu$ -a.e. by the dominated convergence theorem. Thus

$$(5) \quad \lim_{\varepsilon \rightarrow 0} I = 2C^{C^{\mu} d\mu}(z) \text{ if } z \in E.$$

It is left to show that, since  $z \in F$ ,

$$(6) \quad \lim_{\varepsilon \rightarrow 0} I = [C^{\mu}(z)]^2.$$

Since  $z \in F$ , the following integral converges absolutely:

$$\phi_{\varepsilon}(t, z) := \int_{|\zeta-t|>\varepsilon} \frac{d\mu(\zeta)}{\zeta-z}.$$

Also

$$I = \int \phi_{\varepsilon}(t, z) \frac{1}{t-z} d\mu(t).$$

Since the point  $z$  is fixed in  $F$ , we have that  $\frac{1}{|\zeta-z|} \in L^1(|\mu|)$ , and therefore  $\int_A \frac{1}{|\zeta-z|} d|\mu|(\zeta)$  is small if  $|\mu|(A)$  is small. Denoting the disc centered at  $t$  and of radius  $\varepsilon$  by  $B(t, \varepsilon)$  we notice that

- 1)  $\phi_{\varepsilon}(t, z) = \int_{\mathbb{C}} \frac{d\mu(\zeta)}{\zeta-z} - \int_{B(t, \varepsilon)} \frac{d\mu(\zeta)}{\zeta-z},$
- 2)  $\lim_{\varepsilon \rightarrow 0} |\mu|(B(t, \varepsilon)) = 0$   
uniformly in  $t$ . Otherwise  $\mu$  would have an atom.

We conclude that, as  $\varepsilon \rightarrow 0$ , the functions  $\phi_{\varepsilon}(t, z)$  converge uniformly in  $t \in \mathbb{C}$  to  $\phi(z) = \int \frac{d\mu(\zeta)}{\zeta-z}$ . Hence for any  $z \in F$  and any  $t \in \mathbb{C} \setminus z$

- 3)  $\frac{\phi_{\varepsilon}(t, z)}{t-z} \rightarrow \frac{\phi(z)}{t-z},$  as  $\varepsilon \rightarrow 0$ .

Since  $\phi_{\varepsilon}(t, z)$  converge uniformly and  $z \in F$ ,

$$\int \phi_{\varepsilon}(t, z) \frac{1}{t-z} d\mu(t) \rightarrow \phi(z) \int \frac{d\mu(t)}{t-z} = [C^{\mu}(z)]^2.$$

We have verified (6).

Combining (5) and (6) we conclude that for  $z \in E \cap F$  (so for  $m_2$ -a.e.  $z \in \mathbb{C}$ ) we have

$$(7) \quad 2C^{C^\mu d\mu}(z) = 2 \int \frac{C^\mu(t) d\mu(t)}{t-z} = \lim_{\varepsilon \rightarrow 0} I = [C^\mu(z)]^2$$

for  $m_2$ -a.e. point  $z \in \mathbb{C}$ .

This formula is true as long as  $C_*^\mu \in L^1(|\mu|)$  and the principal value  $C^\mu$  exists  $\mu$ -a.e.  $\square$

To deduce Theorem 2.1 suppose that  $C^\mu$  vanishes  $\mu$ -a.e. Then the left-hand side in (7) is zero for  $m_2$ -a.e. point  $z$ . The same must hold for  $[C^\mu(z)]^2$ . But if  $C^\mu(z) = 0$  for Lebesgue-a.e. point  $z \in \mathbb{C}$  then  $\mu = 0$ , see for example [5]. Theorem 2.1 is completely proved.

*Remark.* In the statement of Theorem 2.2 the condition  $C_*^\mu \in L^1(|\mu|)$  can be replaced with the condition that  $C_\varepsilon^\mu$  converge in  $L^1(|\mu|)$ . The proof would have to be changed as follows.

Like in the above proof one can show that at Lebesgue-a.e. point  $z$

$$(8) \quad \lim_{\varepsilon \rightarrow 0} I = [C^\mu(z)]^2.$$

The relation

$$I = 2 \int \frac{C_\varepsilon^\mu(t) d\mu(t)}{t-z}$$

for a.e.  $z$  can also be established as before. Since  $C_\varepsilon^\mu$  converges in  $L^1(|\mu|)$ , the last integral converges to  $C^{C^\mu d\mu}(z)$  in the “weak”  $L^2(dx dy)$ , which concludes the proof.

Hence we arrive at the following version of Theorem 2.1:

**Theorem 2.3.** *Let  $\mu \in M_c$ . Assume that  $C_\varepsilon^\mu \rightarrow 0$  in  $L^1(|\mu|)$ . Then  $\mu \equiv 0$ .*

This version has the following corollary:

**Corollary 2.4** ([15]). *Let  $\mu \in M$  be a measure of linear growth and finite Menger curvature. If  $C^\mu = 0$  at  $\mu$ -a.e. point then  $\mu \equiv 0$ .*

*Proof:* The conditions on  $\mu$  imply that the  $L^2(|\mu|)$ -norms of the functions  $C_\varepsilon^\mu$  are uniformly bounded, see for instance [11]. Since  $C_\varepsilon^\mu$  also converge  $\mu$ -a.e., they must converge in  $L^1(|\mu|)$ .  $\square$

*Remark.* As was mentioned in the Introduction, Corollary 2.4 also follows from Theorem 2.1. However, the above version of the argument allows one to obtain it without using the additional results of [12], [14] on the maximal function.

We also obtain the following statement on the injectivity of any bounded planar Cauchy transform. As usual, we say that the Cauchy transform is bounded in  $L^2(\mu)$  if the functions  $C_\varepsilon^{f d\mu}$  are bounded, uniformly with respect to  $\varepsilon$ , in  $L^2(\mu)$ -norm for any  $f \in L^2(\mu)$ . If  $C^\mu$  is bounded, then  $C_\varepsilon^{f d\mu}$  converge  $\mu$ -a.e. as  $\varepsilon \rightarrow 0$  and the image  $C^{f d\mu}$  exists in a regular sense as a function in  $L^2(\mu)$ , see [14].

**Corollary 2.5.** *Let  $\mu \in M$  be a positive measure. If  $C^\mu$  is bounded in  $L^2(\mu)$  then it is injective (has a trivial kernel).*

*Proof:* Suppose that there is  $f \in L^2(\mu)$  such that  $C^{f d\mu} = 0$  at  $\mu$ -a.e. point. Since both  $f$  and  $C_*^{f d\mu}$  are in  $L^2(\mu)$ ,  $C_*^{f d\mu}$  is in  $L^1(|f| d\mu)$ . Hence  $f$  is a zero-function by Theorem 2.1.  $\square$

*Remark.* We have actually obtained a slightly stronger statement: If  $C^\mu$  is bounded in  $L^2(\mu)$  then for any  $f \in L^2(\mu)$  the functions  $f$  and  $C^{f d\mu}$  cannot have disjoint essential supports, i.e. the product  $f C^{f d\mu}$  cannot equal to 0 at  $\mu$ -a.e. point.

In the rest of this section we will discuss what other kernels could replace the Cauchy kernel in the statement of Theorem 2.1.

If  $K(x)$  is a complex-valued function in  $\mathbb{R}^n$ , bounded outside of any neighborhood of the origin, and  $\mu$  is a finite measure on  $\mathbb{R}^n$ , one can define  $K^\mu$  and  $K_*^\mu$  in the same way as  $C^\mu$  and  $C_*^\mu$  were defined in the Introduction.

The proof of Theorem 2.2 relied on the fact that the Cauchy kernel  $K(z) = 1/z$  is odd, satisfies the symmetry condition (3), i.e.

$$(9) \quad K(x-y)K(y-z) + K(y-z)K(z-x) + K(z-x)K(x-y) \equiv 0,$$

and is summable as a function of  $z$  for any  $t$  with respect to Lebesgue measure. Kernels satisfying other symmetry conditions instead of (9) may lead to formulas similar to Theorem 2.2 that could still yield Theorem 2.1.

Here is a different example. It shows that much less symmetry can be required from the kernel if the measure is positive.

**Theorem 2.6.** *Let  $\mu$  be a positive measure in  $\mathbb{R}^n$ . Suppose that the real kernel  $K(x)$  satisfies the following properties:*

- 1)  $K(-x) = -K(x)$  for any  $x \in \mathbb{R}^n$ ;
- 2)  $K(x) > 0$  for any  $x$  from the half-space  $\mathbb{R}_+^n = \{x = (x_1, x_2, \dots, x_n) \mid x_1 > 0\}$ .

*If  $K_*^\mu \in L^1(\mu)$  and  $K^\mu(x) = 0$  for  $\mu$ -a.e.  $x$  then  $\mu \equiv 0$ .*

Note that real and imaginary parts of the Cauchy kernel, Riesz kernels in  $\mathbb{R}^n$ , as well as many other standard kernels satisfy the conditions of the theorem.

We will need the following

**Lemma 2.7.** *Let  $K$  be an odd kernel. and let  $\mu, \nu \in M$ . Then*

$$(10) \quad \int K_\varepsilon^\mu(z) d\nu(z) = - \int K_\varepsilon^\nu(z) d\mu(z)$$

for any  $\varepsilon > 0$ .

*Suppose that  $K_*^\mu \in L^1(|\nu|)$ . If  $K^\mu(z)$  exists  $\nu$ -a.e. then*

$$\int K^\mu(z) d\nu(z) = - \lim_{\varepsilon \rightarrow 0} \int K_\varepsilon^\nu(z) d\mu(z).$$

*In particular, suppose that both  $K_*^\mu \in L^1(|\nu|)$  and  $K_*^\nu \in L^1(|\mu|)$ . If  $K^\mu(z)$  exists  $\nu$ -a.e. and  $K^\nu(z)$  exists  $\mu$ -a.e. then*

$$\int K^\mu(z) d\nu(z) = \int K^\nu(z) d\mu(z).$$

*Proof:* Since  $K$  is odd, the first equation can be obtained simply by changing the order of integration. The second and third equations now follow from the dominated convergence theorem.  $\square$

*Proof of Theorem 2.6:* There exists a hyperplane  $\{x_1 = c\}$  in  $\mathbb{R}^n$  such that  $\mu(\{x_1 = c\}) = 0$  but both  $\mu(\{x_1 > c\})$  and  $\mu(\{x_1 < c\})$  are non-zero. Denote by  $\nu$  and  $\eta$  the restrictions of  $\mu$  onto  $\{x_1 > c\}$  and  $\{x_1 < c\}$  respectively. Then

$$\int K_\varepsilon^\nu(z) d\mu(z) = \int K_\varepsilon^\nu(z) d\nu(z) + \int K_\varepsilon^\nu(z) d\eta(z).$$

The first integral on the right-hand side is 0 because of the oddness of  $K$  (apply the first equation in the last lemma with  $\mu = \nu$ ). The second condition on  $K$  and the positivity of the measure imply that the second integral is positive and increases as  $\varepsilon \rightarrow 0$ . Therefore  $\int K_\varepsilon^\nu(z) d\mu(z)$  cannot tend to zero. This contradicts the fact that  $K^\mu = 0$ ,  $\nu$ -a.e. and the second equation from the last lemma.  $\square$



### 3. Sets of finite perimeter

In this section we give another example of an application of Theorem 2.2. It involves the notion of a set of finite perimeter introduced by De Giorgi in the 50's, see [4]. We say that a set  $G \subset \mathbb{R}^2$  has finite perimeter (in the sense of De Giorgi) if the distributional partial derivatives of its characteristic function  $\chi_G$  are finite measures. Such sets have structural theorems. For example, if  $G$  is such a set then the vector measure  $\nabla \chi_G$  is carried by a set  $E$ , rectifiable in the sense of Besicovitch, i.e. a subset of a countable union of  $C^1$  curves and an  $\mathcal{H}^1$ -null set, where  $\mathcal{H}^1$  is the one-dimensional Hausdorff measure. Also the measure  $\nabla \chi_G$  is absolutely continuous with respect to  $\mathcal{H}^1$  restricted to  $E$  and its Radon-Nikodym derivative is a unit normal vector  $\mathcal{H}^1$ -a.e. At  $\mathcal{H}^1$ -almost all points of  $E$  the function  $\chi_G$  has approximate "one-sided" limits. For more details we refer the reader to [4].

The general question we consider can be formulated as follows: What can be said about  $\mu$  if  $C^\mu$  coincides at  $\mu$ -a.e. point with a "good" function  $f$ ? To avoid certain technical details, all measures in this section are compactly supported. Furthermore, we will only discuss the two simplest choices of  $f$ . As we will see, even in such elementary situations Theorem 2.2 yields interesting consequences.

As usual, when we say that  $C^\mu = f$  at  $\mu$ -a.e. point, we imply that the principal value exists  $\mu$ -almost everywhere.

**Theorem 3.1.** *Let  $\mu \in M_c$  be compactly supported. Assume that  $C^\mu(z) = 1$ ,  $\mu$ -almost everywhere and  $C_*^\mu \in L^1(|\mu|)$ . Then  $\mu = \bar{\partial} \chi_G$ , where  $G$  is a set of finite perimeter. In particular,  $\mu$  is carried by a set  $E$ ,  $\mathcal{H}^1(E) < \infty$ , rectifiable in the sense of Besicovitch, and  $\mu$  is absolutely continuous with respect to the restriction of  $\mathcal{H}^1$  to  $E$ .*

*Remark.* The most natural example of such a measure is  $dz$  on a  $C^1$  closed curve. The theorem says that, by the structural results of De Giorgi, this is basically the full answer.

*Proof:* By Theorem 2.2 we get that for Lebesgue-almost every point in  $\mathbb{C}$

$$(11) \quad [C^\mu(z)]^2 = 2C^\mu(z).$$

In other words for  $m_2$ -a.e. point  $z$  we have  $C^\mu(z) = 0$  or  $= 2$ . Let  $G$  denote the set where  $C^\mu(z) = 2$ . Since the Cauchy transform of any compactly supported finite measure must tend to zero at infinity, this set is bounded. Consider the following equality

$$\chi_G = C^{\mu/2},$$

understood in the sense that the two functions are equal as distributions. Taking distributional derivatives on both sides we obtain

$$\bar{\partial}\chi_G = \mu/2 \quad \text{and} \quad \partial\bar{\chi}_G = \bar{\mu}/2.$$

Hence  $G$  has finite perimeter and the rest of the statement follows from the results of [4].  $\square$

We say that a set  $G$  has locally finite perimeter (in the sense of De Giorgi) if the distributional derivatives of  $\chi_G$  are locally finite measures. Our second application is the following

**Theorem 3.2.** *Let  $\mu \in M_c$  be compactly supported. Assume that  $C^\mu(z) = z$ ,  $\mu$ -almost everywhere and  $C_*^\mu \in L^1(|\mu|)$ . If  $\mu(\mathbb{C}) = 0$  then  $\mu = 2z\bar{\partial}\chi_G$ , where  $G$  is a set with locally finite perimeter. Whether  $\mu(\mathbb{C}) = 0$  or not,  $\mu$  is carried by a set  $E$ ,  $\mathcal{H}^1(E) < \infty$ , which is a rectifiable set in the sense of Besicovitch, and  $\mu$  is absolutely continuous with respect to the restriction of  $\mathcal{H}^1$  to  $E$ .*

*Remark.* The most natural example of such a measure is  $z dz$  on a  $C^1$  closed curve. Our statement shows that this is basically one-half of the answer. The other half is given by  $\sqrt{z^2 - c} dz$  as will be seen from the proof.

*Proof:* Again, from Theorem 2.2 we get that for Lebesgue-almost every point in  $\mathbb{C}$

$$(12) \quad [C^\mu(z)]^2 = 2C^\zeta d\mu(\zeta)(z).$$

Notice that

$$C^\zeta d\mu(\zeta)(z) = \int \frac{\zeta}{\zeta - z} d\mu(\zeta) = \mu(\mathbb{C}) + zC^\mu(z)$$

and we get a quadratic equation

$$[C^\mu(z)]^2 = 2zC^\mu(z) - p,$$

where  $p := -2\mu(\mathbb{C})$ .

*First case:*  $p = 0$ . Here we get

$$[C^\mu(z)]^2 = 2zC^\mu(z).$$

We conclude that  $C^\mu(z) = 0$  or  $z$  for Lebesgue-a.e. point  $z \in \mathbb{C}$ .

Again a bounded set  $G$  appears on which

$$C^\mu = 2z\chi_G(z)$$

in terms of distributions. Therefore

$$\bar{\partial}\chi_G = d\mu/2z,$$

and the right hand side is a finite measure on any compact set avoiding the origin. Therefore,  $G$  is a (locally) De Giorgi set.

Let us consider the case  $p \neq 0$ . For simplicity we assume  $p = 1$ , other  $p$ 's are treated in the same way. Following a suggestion by the referee let us mention that the argument below utilizes some of the properties of the Joukowski function. If  $p = 1$  then we have to solve the quadratic equation

$$C^\mu(z)^2 - 2zC^\mu(z) + 1 = 0$$

for Lebesgue-a.e. point in  $\mathbb{C}$ . Let us make the slit  $[-1, 1]$  and consider two holomorphic functions in  $\mathbb{C} \setminus [-1, 1]$

$$r_1(z) = z - \sqrt{z^2 - 1}, \quad r_2(z) = z + \sqrt{z^2 - 1},$$

where the branch of the square root is chosen so that

$$r_1(z) \rightarrow 0, \quad z \rightarrow \infty.$$

The above equation for  $C^\mu$  implies that there exist disjoint sets  $E_1$  and  $E_2$ ,  $m_2(\mathbb{C} \setminus E_1 \cup E_2) = 0$ , such that

$$\begin{aligned} z \in E_1 &\Rightarrow C^\mu(z) = r_1(z), \\ z \in E_2 &\Rightarrow C^\mu(z) = r_2(z). \end{aligned}$$

Obviously it is  $E_1$  that contains a neighborhood of infinity. The function  $z - \sqrt{z^2 - 1}$  outside of  $[-1, 1]$  can be written as  $C^{\mu_0}(z)$  where  $d\mu_0(x) = \frac{1}{\pi}\sqrt{1-x^2}dx$ . Consider  $\nu = \mu - \mu_0$ . Then

$$\begin{aligned} z \in E_1 &\Rightarrow C^\nu(z) = 0, \\ z \in E_2 &\Rightarrow C^\nu(z) = 2\sqrt{z^2 - 1} := R(z). \end{aligned}$$

Therefore,

$$(13) \quad C^\nu(z) = R(z)\chi_{E_2}.$$

Notice that if  $R$  was analytic in an open domain compactly containing  $E_2$  we would conclude from the previous equality that

$$\nu = R(z)\bar{\partial}\chi_{E_2}.$$

If, in addition,  $|R|$  was bounded away from zero on  $E_2$ , we would obtain that  $\bar{\partial}\chi_{E_2}$  and  $\partial\chi_{E_2}$  are measures of finite variation, and hence  $E_2$  is a set of finite perimeter. Notice that our  $R(z) = 2\sqrt{z^2 - 1}$  is analytic

in  $O := \mathbb{C} \setminus [-1, 1]$  and is nowhere zero. We conclude that  $E_2$  is a set of locally finite perimeter. More precisely we establish the following claim:

For every open disk  $V \subset O$  the set  $O \cap E_2$  has finite perimeter.

Indeed, let  $W$  be a disk compactly containing  $V$ ,  $W \subset O$ . Let  $\psi$  be a smooth function, supported in  $W$ ,  $\psi|_V = 1$ . Multiply (13) by  $\psi$  and take a distributional derivative (against smooth functions supported in  $V$ ). Then we get (using the fact that  $R$  is holomorphic on  $V$ )

$$\nu|_V = \bar{\partial}(\psi R \chi_{E_2 \cap V})|_V = \bar{\partial}(R \chi_{E_2 \cap V})|_V = R \bar{\partial}(\chi_{E_2 \cap V})|_V.$$

We conclude immediately that  $E_2 \cap V$  is a set of finite perimeter. Therefore,  $E_2 \cap D$  is a set of finite perimeter, where  $D$  is a domain whose closure is contained compactly in  $O$ .

Recalling that  $\mu = \nu + \mu_0$  finishes the proof.  $\square$

*Remark 3.3.* It is interesting to note that if  $\mu$  is the measure from the statement of the theorem then one of the connected components of  $\text{supp } \mu$  must contain both roots of the equation  $z^2 + 2\mu(\mathbb{C}) = 0$ . Indeed, since  $C^\mu$  is analytic in the complement of  $\text{supp } \mu$  and satisfies the quadratic equation used in the proof,  $\text{supp } \mu$  must contain the slit between these two points.

We conclude this section with the following examples of measures  $\mu$  whose Cauchy transform coincides with  $z$  at  $\mu$ -a.e. point.

**Examples.** 1) Let  $\Omega$  be an open domain with smooth boundary  $\Gamma$ . Suppose that  $[-1, 1] \subset \Omega$ . Let  $\{D_j\}_{j=1}^\infty$  be smoothly bounded disjoint domains in  $\mathcal{O} := \Omega \setminus [-1, 1]$ ,  $\gamma_j = \partial D_j$ . Assume

$$(14) \quad \sum_j \mathcal{H}^1(\gamma_j) < \infty.$$

Let  $R(z)$  be an analytic branch of  $2\sqrt{z^2 - 1}$  in  $\mathcal{O}$ . Consider the measure  $\nu$  on  $\Gamma \cup (\cup_j \gamma_j) \cup [-1, 1]$  defined as

$$\nu = R(z) dz|_\Gamma - R(z) dz|_{\cup_j \gamma_j} - \frac{1}{\pi} \sqrt{1 - x^2} dx|_{[-1, 1]}.$$

Then

$$C^\nu(z) = \begin{cases} 0 & \text{if } z \in \mathbb{C} \setminus \bar{\mathcal{O}}, \\ 0 & \text{if } z \in \cup_j D_j, \\ R(z) & \text{if } z \in \mathcal{O} \setminus \cup_j \bar{D}_j. \end{cases}$$

Recall that  $R(z) = z + \sqrt{z^2 - 1} - (z - \sqrt{z^2 - 1})$  and that  $C^{\mu_0}(z) = z - \sqrt{z^2 - 1}$  for  $\mu_0 = \frac{1}{\pi} \sqrt{1 - x^2} dx|_{[-1,1]}$ . We conclude that for  $\mu = \nu + \mu_0$  one has

$$C^\mu(z) = \begin{cases} z - \sqrt{z^2 - 1} & \text{if } z \in \mathbb{C} \setminus \bar{\mathcal{O}}, \\ z - \sqrt{z^2 - 1} & \text{if } z \in \cup_j D_j, \\ z + \sqrt{z^2 - 1} & \text{if } z \in \mathcal{O} \setminus \cup_j \bar{D}_j. \end{cases}$$

2) The second example is exactly the same as the first one but  $D_{j,k} = B(x_{j,k}, \frac{1}{10j^2})$ ,  $x_{j,k} = 2 + \frac{1}{j} e^{\frac{2\pi i k}{j}}$ ,  $1 \leq k \leq j$ ,  $j = 1, 2, 3, \dots$ . Here the assumption (14) fails. But  $\nu$ , defined as above, will still be a measure of finite variation (and so will  $\mu$ ):  $|\nu|(\mathbb{C}) \leq C \sum_j \frac{1}{j^{3/2}}$ .

In both examples  $C^\mu(z) = z$  for  $\mu$ -a.e.  $z$ .

#### 4. Asymptotic behavior near the zero-set of $C^\mu$

In this section we take a slightly different approach. We study asymptotic properties of measures near the sets where the Cauchy transform vanishes. Theorem 4.2 below shows that near the density points of such sets the measure must display a certain “irregular” asymptotic behavior.

As was mentioned in the Introduction, one of the results of [15] says that an absolutely continuous planar measure cannot be reflectionless. This result is not implied by our Theorem 2.1 because an absolutely continuous measure may not have a summable Cauchy maximal function. It is, however, implied by Theorem 4.2, see Corollary 4.4 below.

When estimating Cauchy integrals one often uses an elementary observation that the difference of any two Cauchy kernels  $1/(z-a) - 1/(z-b)$  can be estimated as  $O(|z|^{-2})$  near infinity. To obtain higher order of decay one may consider higher order differences. Here we will utilize the following estimate of that kind, which can be verified through simple calculations.

**Lemma 4.1.** *Let  $a, b, c \in B(0, r)$  be different points,  $|a - b| > r$ . Then there exist constants  $A, B \in \mathbb{C}$  such that  $|A|, |B| < 2$*

$$(15) \quad \left| \frac{A}{z-a} + \frac{B}{z-b} - \frac{1}{z-c} \right| < \frac{Cr^2}{|z|^3}$$

outside of  $B(0, 2r)$ .

(Namely,  $A = \frac{b-c}{b-a}$ ,  $B = \frac{a-c}{a-b}$ .)

If  $\mu \in M$  consider one of its Riesz transforms in  $\mathbb{R}^3$ ,  $R_1\mu(x, y, z)$ , defined as

$$R_1\mu(x, y, z) = \int \frac{z}{|(u, v, 0) - (x, y, z)|^3} d\mu(u + iv).$$

This transform is the planar analogue of the Poisson transform. In particular,

$$\lim_{z \rightarrow 0+} R_1\mu(x, y, z) = \frac{d\mu}{dm_2}(x + iy)$$

for all points  $w = x + iy \in \mathbb{C}$  where the Radon derivative

$$\frac{d\mu}{dm_2}(w) = \lim_{r \rightarrow 0+} \frac{\mu(B(w, r))}{|B(w, r)|}$$

exists.

For measures on the line or on the circle their Poisson integrals and Radon derivatives (with respect to the one-dimensional Lebesgue measure) are very much related but not always equivalent. When the asymptotics of the Poisson integral and the ratio from the definition of the Radon derivative are different near a certain point it usually means that the measure is “irregular” near that point. It is not difficult to show that if  $\mu$  is absolutely continuous then at a Lebesgue point of its density function the Radon derivative of  $\mu$  and the Poisson integral of  $|\mu|$  (or  $R_1|\mu|$  if  $n > 1$ ) behave equivalently. Even for singular measures on the circle, if a measure possesses a certain symmetry near a point, then the same equivalent behavior takes place, as follows for instance from [1, Lemma 4.1]. In fact, it is not easy to construct a measure so that its Poisson integral and Radon derivative behaved differently near a large set of points. The same can be said about the Riesz transform and the Radon derivative. Thus one may interpret our next result as an evidence that, for a planar measure  $\mu$ , most points where  $C^\mu = 0$  are “irregular”.

**Theorem 4.2.** *Let  $\mu \in M$  and let  $w = x + iy$  be a point of density (with respect to  $m_2$ ) of the set  $E = \{C^\mu = 0\}$ . Then*

$$\frac{\mu(B(w, r))}{\pi r^2} = o(R_1|\mu|(x, y, r)) \quad \text{as } r \rightarrow 0+.$$

In view of the above discussion this implies

**Corollary 4.3.** *If  $w$  is a point of density of the set  $E = \{C^\mu = 0\}$ , such that the Radon derivative  $d|\mu|/dm_2(w)$  exists and is non-zero, then*

$$(16) \quad \mu(B(w, r)) = o(|\mu|(B(w, r))) \quad \text{as } r \rightarrow 0+$$

and  $d\mu/dm_2(w) = 0$ .

Since  $m_2$ -almost every point of a set is a density point of that set, we also obtain the following version of the result from [15]:

**Corollary 4.4.** *The set  $E = \{C^\mu = 0\}$  has measure zero with respect to the absolutely continuous component of  $\mu$ .*

*Proof of Theorem 4.2:* Without loss of generality  $w = 0$ . Choose a  $C_0^\infty$  test-function  $\phi$  supported in  $B := B(0, r)$ , and such that  $0 \leq \phi \leq D/r^2$ ,  $|\nabla \phi| \leq A/r^3$  and  $\int_{\mathbb{C}} \phi dm_2 = 1$ . Denote the complement of  $E$  by  $E^c$ . Then

$$(17) \quad \begin{aligned} \int \phi d\mu &= \langle \phi, \bar{\partial} C^\mu \rangle = \langle \bar{\partial} \phi, C^\mu \rangle \\ &= \langle \chi_{E^c} \bar{\partial} \phi, C^\mu \rangle = \int \left( \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} \right) d\mu(\zeta). \end{aligned}$$

All we need is to show that the last integral is small. Then, since the first integral in (17) is similar to the right-hand side of (16), this will complete the proof. The main idea for the rest of the proof is to make the function  $F(\zeta) = \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z}$  “small” by subtracting a linear combination of Cauchy kernels corresponding to points from  $E$ , which will not change its integral with respect to  $\mu$ .

Namely, let  $a, b \in B(0, r) \cap E$  be any two points such that  $|a - b| > r$ . By the previous lemma for any  $z \in B(0, r)$  there exist constants  $A = A(z)$ ,  $B = B(z)$ , of modulus at most 2, such that (15) holds with  $c = z$ . Integrating (15) with respect to  $\chi_{E^c} \bar{\partial} \phi dm_2(z)$  we obtain that

$$\left| \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} - \frac{A^*}{\zeta - a} - \frac{B^*}{\zeta - b} \right| < C \frac{\varepsilon(r)r}{|\zeta|^3}$$

outside of  $B(0, 2r)$  for some constants  $A^*, B^*$ , where  $\varepsilon(r) = |B(0, r) \cap E^c|/r^2 = o(1)$  as  $r \rightarrow 0$ . The constants satisfy  $|A^*|, |B^*| < 2 \frac{\varepsilon(r)}{r}$ .

Notice that if  $w \in E$  then  $\int \frac{1}{\zeta - w} d\mu = 0$  by the definition of the set  $E$ . Hence, since  $a, b \in E$ ,

$$\begin{aligned} \int \left( \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} \right) d\mu(\zeta) &= \int \left( \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} - \frac{A^*}{\zeta - a} - \frac{B^*}{\zeta - b} \right) d\mu(\zeta) \\ &= \int_{B(0, 2r)} + \int_{\mathbb{C} \setminus B(0, 2r)} = I_1 + I_2. \end{aligned}$$

For  $I_2$  we now have

$$\begin{aligned} & \left| \int_{\mathbb{C} \setminus B(0, 2r)} \left( \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} - \frac{A^*}{\zeta - a} - \frac{B^*}{\zeta - b} \right) d\mu(\zeta) \right| \\ & \leq C \int_{\mathbb{C} \setminus B(0, 2r)} \frac{\varepsilon(r)r}{|\zeta|^3} d|\mu|(\zeta) \leq C\varepsilon(r)R_1|\mu|(0, 0, r). \end{aligned}$$

In  $I_1$  we estimate each summand separately. First,

$$\begin{aligned} & \left| \int_{B(0, 2r)} \left( \int \frac{\chi_{E^c} \bar{\partial} \phi dm_2(z)}{\zeta - z} \right) d\mu(\zeta) \right| \\ & \leq \int_{B(0, 2r)} \frac{D}{r^3} \int \frac{1}{|\zeta - z|} \chi_{E^c} dm_2(z) d|\mu|(\zeta) \\ & \leq C \frac{\sqrt{\varepsilon(r)}}{r^2} |\mu|(B(0, 2r)) \leq C\sqrt{\varepsilon(r)}R_1|\mu|(0, 0, r). \end{aligned}$$

To estimate the second and third summands of  $I_1$ , recall that the only restriction on the choice of  $a, b \in B(0, r) \cap E$  was that  $|a - b| > r$ . This condition will be satisfied, for instance, if  $a \in B_1 = B(-\frac{5}{6}r, \frac{1}{6}r)$  and  $b \in B_2 = B(\frac{5}{6}r, \frac{1}{6}r)$ . If we average the modulus of the second summand over all choices of  $a \in B_1 \cap E$ , recalling that  $A^* = A^*(a)$  always satisfies  $|A^*| \leq 2\frac{\varepsilon(r)}{r}$ , we get

$$\begin{aligned} & \frac{1}{|B_1 \cap E|} \int_{B_1 \cap E} \left| \int_{B(0, 2r)} \frac{A^*(a)}{\zeta - a} d\mu(\zeta) \right| dm_2(a) \\ & \leq \frac{1}{|B_1 \cap E|} \int_{B(0, 2r)} \int_{B_1 \cap E} \frac{|A^*(a)|}{|\zeta - a|} dm_2(a) d|\mu|(\zeta) \\ & \leq C \frac{1}{r^2} \frac{\varepsilon(r)}{r} r |\mu|(B(0, 2r)) \leq C\varepsilon(r)R_1|\mu|(0, 0, r). \end{aligned}$$

It is left to choose  $a \in B_1 \cap E$  for which the modulus is no greater than its average. The same can be done for  $b$ . The proof is finished.  $\square$

## 5. Reflectionless measures and Combs

As was mentioned in the Introduction, following [2], we will call a non-trivial continuous finite measure  $\mu \in M(\mathbb{C})$  *reflectionless* if  $C^\mu(z) = 0$  at  $\mu$ -a.e. point  $z$ .



Perhaps the simplest example of a reflectionless measure is the measure  $\mu = \frac{1}{\pi}(1-x^2)^{-1/2} dx$  on  $[-1, 1]$ , the harmonic measure of  $\mathbb{C} \setminus [-1, 1]$  corresponding to infinity. The fact that  $\mu$  is reflectionless can be verified through routine calculations or via the conformal map interpretation of the harmonic measure. It will also follow from a more general Theorem 5.4 below.

At the same time, since  $C_*^\mu \asymp (1-x^2)^{-1/2}$  on  $[-1, 1]$ , this simple example complements the statement of Theorem 2.1. Since the function  $(1-x^2)^{-1/2}$  belongs to the “weak”  $L^1(|\mu|)$ , the summability condition for the Cauchy maximal function proves to be exact in its scale.

In the rest of this section we discuss further examples and properties of positive reflectionless measures on the line.

Let us recall that functions holomorphic in the upper half plane  $\mathbb{C}_+$  and mapping it to itself (having non-negative imaginary part) are called Nevanlinna functions. Let  $M_+(\mathbb{R})$  denote the class of finite positive measures compactly supported on  $\mathbb{R}$ . The function  $f$  is a Nevanlinna function if and only if it has a form

$$f(z) = az + b + \int_{\mathbb{R}} \left[ \frac{1}{t-z} - \frac{t}{t^2+1} \right] d\rho(t),$$

where  $\rho$  is a positive measure on  $\mathbb{R}$  such that  $\int \frac{d\rho(t)}{t^2+1} < \infty$ ,  $a > 0, b \in \mathbb{R}$  are constants. If the representing measure is from  $M_+(\mathbb{R})$  and  $f(\infty) = 0$ , the formula becomes simpler:  $f(z) = \int \frac{d\mu(x)}{x-z}$ .

**Definition.** A simply connected domain  $\mathcal{O}$  is *comb-like* if it is a subset of a half-strip  $\{w : \Im w \in (0, \pi), \Re w > q\}$ , for some  $q \in \mathbb{R}$ , contains another half-strip  $\{w : \Im w \in (0, \pi), \Re w > r\}$  for some  $r \in \mathbb{R}$  and has the property that

- (18) for any  $w_0 = u_0 + iv_0 \in \mathcal{O}$   
the whole ray  $\{w = u + iv_0, u \geq u_0\}$  lies in  $\mathcal{O}$ .

If in addition  $\mathcal{H}^1(\partial\mathcal{O} \cap B(0, R)) < \infty$  for all finite  $R$ , we say that  $\mathcal{O}$  is a *rectifiable comb-like* domain.

Let  $\mathcal{O}$  be a *rectifiable comb-like* domain,  $\Gamma = \partial\mathcal{O}$ . Then by the Besicovitch theory we know that for  $\mathcal{H}^1$ -a.e. point  $w \in \Gamma$  there exists an approximate tangent line to  $\Gamma$ , see [10] for details. We wish to consider

rectifiable comb-like domains satisfying the following geometric property:

(19) for a.e.  $w \in \Gamma$

the approximate tangent line is either vertical or horizontal.

It is not difficult to verify that for any conformal map  $F: \mathbb{C}_+ \rightarrow \mathcal{O}$ ,  $\mathcal{O}$  is comblike if and only if  $F'$  is a Cauchy potential of  $\mu \in M_+(\mathbb{R})$ :  $F'(z) = \int \frac{d\mu(x)}{x-z}$ . It is, therefore, natural to ask the following

**Question.** Which comb-like domains correspond to reflectionless measures  $\mu \in M_+(\mathbb{R})$ ?

An answer would give a geometric description of reflectionless measures from  $M_+(\mathbb{R})$ . If, in addition, a comb-like domain is rectifiable, then the answer is given by

**Theorem 5.1.** 1) Rectifiable comb-like domains correspond exactly to those measures  $\mu \in M_+(\mathbb{R})$  that are absolutely continuous with respect to  $dx$  and satisfy

$$(20) \quad \int \frac{d\mu(x)}{x-z} \in H_{\text{loc}}^1(\mathbb{C}_+).$$

2) An absolutely continuous measure satisfying (20) is reflectionless if and only if the corresponding comb-like domain has the property (19).

*Remarks.*

- 1) Of course not every comb-like domain gives rise to a reflectionless measure from  $M_+(\mathbb{R})$ . Just take any comb-like domain which appears as  $F(\mathbb{C}_+)$ , where  $F = \int^z \int \frac{d\mu(x)}{x-z}$  for a singular  $\mu \in M_+(\mathbb{R})$ . By a result from [8] singular measures cannot be reflectionless.
- 2) On the other hand, even if  $\mu = g(x)dx$  is a reflectionless absolutely continuous measure, the corresponding conformal map  $F = \int^z \int \frac{d\mu(x)}{x-z}: \mathbb{C}_+ \rightarrow \mathcal{O}$  can be onto a non-rectifiable domain.
- 3) For non-rectifiable domains we have no criteria to recognize which ones correspond to reflectionless measures.
- 4) It is well known that the antiderivative of a Nevanlinna function is a conformal map, see for instance [3]. If  $F = \int^z \int \frac{d\mu(x)}{x-z}$ ,  $\mu \in M_+(\mathbb{R})$  then  $\Im F(x)$  is an increasing function on  $\mathbb{R}$  whose derivative in the sense of distributions is  $\mu$ . The image  $F(\mathbb{C}_+)$  lies in the strip  $\{\Im w \in (0, \pi \|\mu\|)\}$ .

Theorem 5.1 will follow from Theorems 5.2 and 5.3 below.

**Theorem 5.2.** *Let  $F$  be a conformal map of  $\mathbb{C}_+$  on a rectifiable comb-like domain  $\mathcal{O}$ . Then  $F(z) = \int^z \int \frac{d\mu(x)}{x-z}$ ,  $\mu \in M_+(\mathbb{R})$ ,  $\mu \ll dx$ . Also  $\int \frac{d\mu(x)}{x-z} \in H_{\text{loc}}^1(\mathbb{C}_+)$ . If in addition  $\mathcal{O}$  satisfies (19) then  $\mu$  is reflectionless.*

*Proof:* Without loss of generality  $\mathcal{O} \subset \{\Re z > 0\}$ . Put  $\Phi = e^F$ . Then the image  $\Phi(\mathcal{O})$  is the subdomain of the complement of the unit half-disk in  $\mathbb{C}_+$  which is the union of rays  $(R(\theta)e^{i\theta}, \infty)$ . Consider the subdomain of the upper half-disk  $D := \{z : 1/z \in \Phi(\mathcal{O})\}$ . Define  $G$  as the smallest open domain containing  $D$  and its reflection  $\bar{D} := \{\bar{z} : z \in D\}$ . Then  $G$  is a star-like domain inside the unit disk. The preimage of  $G \cap \mathbb{R}$  under  $\Phi$  is the union of two infinite rays  $R_1 = [-\infty, a]$ ,  $R_2 = (b, \infty]$ ,  $a < b$ . Therefore, by the reflection principle  $\mathbb{C} \setminus [a, b]$  is mapped conformally (by the extension of  $\Phi$  which we will also denote by  $\Phi$ ) onto the star-like  $G$ .

Since  $\Phi: \mathbb{C}_+ \rightarrow G$ , where  $G$  is star-like, it is well-known that  $\arg \Phi(x + i\delta)$  is an increasing function of  $x$ , see [6].

We conclude that the argument of  $\Phi$  is monotone. Therefore,  $\Im F(x + i\delta)$  is monotone, and so  $\Im f(x + i\delta)$  is positive, where  $f = F'$ . We see that  $f = F'$  is a Nevanlinna function. From the structure of our comb-like domain, we conclude immediately that its representing measure  $\mu$  has compact support, so we are in  $M_+(\mathbb{R})$ . Also, let us prove that  $\mu \ll dx$ . The boundary of our comb is locally rectifiable. So  $f = F'$  belongs locally to the Hardy class  $H^1(\mathbb{C}_+)$ , [13]. Since  $\Im f$  is the Poisson integral of  $\mu$ ,

$$\Im f = P_\mu = \frac{1}{\pi} \int \frac{y}{(x-t)^2 + y^2} d\mu(t),$$

and  $f$  is in  $H^1(\mathbb{C}_+)$  locally, we conclude that  $\mu = \Im f dx$ ,  $\Im f \geq 0$  a.e., [13].

Now suppose that, in addition,  $\mathcal{O} = F(\mathbb{C}_+)$  has the property (19). Let us recall that for a simply connected domain with rectifiable boundary  $\Gamma$  the restriction of the Hausdorff measure  $\mathcal{H}^1|_\Gamma$  is equivalent to the harmonic measure  $\nu$  on  $\mathcal{O}$ . Therefore the tangent lines to  $\Gamma$  are either vertical or horizontal a.e. with respect to  $\nu$ . The measure  $\nu$  is the image of the harmonic measure  $\lambda$  of  $\mathbb{C}_+$  which is equivalent to the Lebesgue measure on the line. We have a conformal map  $F$  (a continuous function up to the boundary of  $\mathbb{C}_+$  because it is an anti-derivative of an  $H_{\text{loc}}^1$ -function) which pushes  $\lambda$  forward to  $\nu$ . Call a point  $w_0 \in \Gamma$  accessible from  $\mathcal{O}$  if there exists a ray  $x_0 + iy$ ,  $0 < y < 1$ , such that  $w_0 = \lim_{y \rightarrow 0} F(x_0 + iy)$ . Almost every point of  $\Gamma$  (w.r. to  $\nu$ ) is accessible from  $\mathcal{O}$ . For  $\nu$ -a.e. accessible  $w_0 \in \Gamma$  where the tangent line is vertical (horizontal) we can say that  $\Re F'(x_0) = 0$  ( $\Im F'(x) = 0$ ). So

$\mathbb{R} = E_1 \cup E_2 \cup E_3$ , where  $|E_3| = 0$ ,  $|E_1 \cap E_2| = 0$ , and  $E_1 = \{x \in \mathbb{R} : \Re F'(x) = 0\}$ ,  $E_2 = \{x \in \mathbb{R} : \Im F'(x) = 0\}$ . We already know that the measure  $\mu = \Im F'(x) dx$  represents  $f(z) = F'(z) = \int_{\mathbb{R} \setminus E_2} \frac{d\mu(t)}{t-z}$ . Notice that  $\int_{\mathbb{R} \setminus E_2} \cdot = \int_{E_1} \cdot$ . But we also know that boundary values exist  $dx$ -almost everywhere, i.e.

$$\lim_{y \rightarrow 0} \Re \int_{E_1} \frac{d\mu(t)}{t - x - iy} = \Re F'(x) = 0$$

for a.e.  $x \in E_1$  and therefore for  $\mu$ -a.e.  $x \in E_1$ . This means (see [13]) that

$$p.v. \int_{\mathbb{R}} \frac{d\mu(x)}{x - z} = 0 \quad \mu\text{-a.e.} \quad \square$$

**Definition.** A simply connected *rectifiable comb-like* domain  $\mathcal{O}$  is called a comb if its “left” boundary consists of countably many horizontal and vertical segments.

A comb is called a straight comb if  $\mathcal{O} = \{w : \Im w \in (0, \pi), \Re w > 0\} \setminus S$ , where the set  $S$  is relatively closed with respect to the strip  $\{w : \Im w \in (0, \pi), \Re w > 0\}$  and is the union of countably many horizontal intervals  $R_n = (iy_n, l_n + iy_n]$ . We require also that

$$\sum_n l_n < \infty.$$

**Example.** Let  $F$  be a conformal map of  $\mathbb{C}_+$  on a comb  $\mathcal{O}$ . By our last theorem  $F'(z) = \int \frac{d\mu(x)}{x-z}$ , where  $\mu \in M_+(\mathbb{R})$  is reflectionless:  $C^\mu(x) = 0$  for  $\mu$ -a.e.  $x$ .

**Definition.** Let  $E$  be a compact subset of the real line. Let  $E$  have positive logarithmic capacity, so Green’s function  $G$  of  $\mathbb{C} \setminus E$  exists. The domain  $\mathbb{C} \setminus E$  is called Widom domain if

$$\sum G(c) < \infty,$$

where the summation goes over all critical points of  $G$  (we assume that  $G$  is a Green’s function with pole at infinity).

**Example.** Let  $E$  be a compact subset of the real line of the positive length. We assume that every point of  $E$  is regular in the sense of Dirichlet for the domain  $\mathbb{C} \setminus E$ , and we also assume that  $\mathbb{C} \setminus E$  is *not* a Widom domain. Such  $E$  exist in abundance. We will see below, that the harmonic measure  $\omega$  of  $\mathbb{C} \setminus E$  (with pole at infinity) is reflectionless. Consider  $F(z) = \int^z \int \frac{d\omega(x)}{z-x}$  for  $z \in \mathbb{C}_+$ . It is easy to see that  $F(z) = G(z) + i\tilde{G}(z) + \text{const.}$ , where  $\tilde{G}$  is the harmonic conjugate of  $G$ . This  $F$  is

a conformal map (see [3]) of  $\mathbb{C}_+$  onto a domain  $D$  lying in the strip  $\{w : \Im w \in (0, \pi)\}$ . It is easy to see that complementary intervals of  $E$  will be mapped by  $F$  onto straight horizontal segments on the boundary of  $D$ . Each finite complementary interval contains exactly one critical point of  $G$ , and clearly the length of the corresponding straight horizontal segment is  $G(c)$  (this follows from the formula  $F(z) = G(z) + i\tilde{G}(z) + \text{const.}$ ).

As the domain  $\mathbb{C} \setminus E$  was *not* a Widom domain, we have that the sum of lengths of abovementioned straight horizontal segments is infinite. So the domain  $D$  is not rectifiable. Therefore the reflectionless property of  $\mu$  alone does not say anything about the rectifiability of the domain, which is the target domain of the conformal map  $F(z) = \int^z \int \frac{d\mu(x)}{z-x}$ .

**Theorem 5.3.** *Let  $\mu$  be an absolutely continuous positive measure on  $\mathbb{R}$  and let  $C^\mu \in H_{\text{loc}}^1(\mathbb{C}_+)$ . Then  $F(z) = \int^z \int \frac{d\mu(x)}{x-z}$  is a conformal map of  $\mathbb{C}_+$  onto a rectifiable comb-like domain  $\mathcal{O}$ . If  $\mu$  is reflectionless then  $\mathcal{O}$  has the property (19).*

*Proof:* Consider  $F(z) = \int^z \int \frac{d\mu(x)}{x-z}$ . Since  $\mu$  is positive, it is a conformal map. If  $\mu$  is such that  $f(z) = C^\mu \in H_{\text{loc}}^1(\mathbb{C}_+)$  then  $F(z) = \int^z f$  maps  $\mathbb{C}_+$  onto a domain with locally rectifiable boundary (see [13]).

If, in addition,  $\mu = \Im f dx$  is reflectionless, then for a.e. point of  $P := \{x \in \mathbb{R} : \Im f(x) > 0\}$  we have  $\Re f(x) = 0$ . The conformal map  $F(z)$  is continuous up to the boundary of  $\mathbb{C}_+$  and its boundary values  $F(x)$  form a (locally) absolutely continuous function,  $F'(x) = f(x)$  a.e. As at almost every point we have either  $\Im F'(x) = 0$  or  $\Re F'(x) = 0$  we conclude that  $\mathcal{O} = F(\mathbb{C}_+)$  has the property (19).  $\square$

We also need the following definition.

**Definition.** A compact subset  $E$  in  $\mathbb{R}$  is called homogeneous if there exist  $r, \delta > 0$  such that for all  $x \in E$ ,  $|E \cap (x-h, x+h)| \geq \delta h$  for all  $h \in (0, r)$ .

**Example.** Let  $E \subset \mathbb{R}$  be a compact set of positive length. Let  $\mu$  be a reflectionless measure supported on  $E$ ,  $\mu = g(x) dx$ . Let in addition  $E$  be a homogeneous set. Then  $F(z) = \int^z \int \frac{d\mu(x)}{x-z}$  is a conformal map from  $\mathbb{C}_+$  on a rectifiable comb-like domain satisfying (19).

*Proof:* The Cauchy integral  $C^{g dx}$  considered in  $\mathbb{C} \setminus E$  will be in the Hardy class  $H^1(\mathbb{C} \setminus E)$ . In fact the reflectionless property of  $g dx$  implies that its limits from  $C_\pm$  will be both integrable with respect to  $dx|_E$ .

Now we use homogeneity of  $E$  and Zinsmeister's theorem [16] to conclude that  $f(z) = C^{g^{dx}}(z)$  is in the usual  $H_{\text{loc}}^1(\mathbb{C})$ . Then the conformal map  $F(z) = \int^z f$  maps  $C_+$  onto a rectifiable subdomain of a strip. We use Theorem 5.3 to get the rest of our example's claims.  $\square$

The simple example of a reflectionless measure mentioned at the beginning of this section, as well as many other explicit examples, are given by our next statement.

**Theorem 5.4.** *Let  $E$  be a compact set of positive length,  $E \subset \mathbb{R}$ . Let  $\omega$  be a harmonic measure of  $\mathbb{C} \setminus E$  with pole at infinity. If  $\omega$  is absolutely continuous, then it is reflectionless.*

**Example.** The simplest comb is a strip  $\{w : \Im w \in (0, \pi), \Re w > 0\}$ . Consider  $F(z) = \log(z + \sqrt{z^2 - 1})$ . It maps conformally  $\mathbb{C}_+$  onto the strip. Its derivative  $f(z) = \frac{1}{\sqrt{z^2 - 1}}$  is  $\frac{1}{\pi} \int \frac{dx}{\sqrt{1-x^2}} \frac{1}{x-z}$  and  $d\mu = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}}$  is the harmonic measure of  $\mathbb{C} \setminus [-1, 1]$ .

*Proof of Theorem 5.4:* We need to show that  $C^\omega = 0$  at  $\omega$ -a.e. point. From our definitions it can be seen, that  $C^\omega$  on the line coincides with the Hilbert transform of  $\omega$ , which in its turn is asymptotically equivalent to the conjugate Poisson transform  $Q^\omega$ . Thus all we need to establish is that

$$\begin{aligned} Q^\omega(x + ih) &= \int_{\mathbb{R}} \frac{x - y}{(x - y)^2 + h^2} d\omega(y) \\ (21) \qquad &= \Re \int \frac{d\omega(y)}{x - ih - y} \rightarrow 0 \quad \text{as } h \rightarrow 0+ \end{aligned}$$

for almost every  $x$ . Instead, we have that the Green's function  $F(x)$  defined as

$$F(x) = \int \log |x - y| d\omega(y) + C_\infty,$$

where  $C_\infty$  is a real constant (Robin's constant), is equal to 0 at every density point of  $E$ , see for example [7]. The idea of the proof is to show that  $Q^\omega(x + i\varepsilon)$  behaves like  $(F(x + \varepsilon) + F(x - \varepsilon))/\varepsilon$  near almost every  $x$ . The technical details are as follows.

Introduce

$$(22) \qquad \phi(y) := \frac{1}{2} \log \frac{|1 - y|}{|1 + y|} + \frac{y}{y^2 + 1},$$

$$\phi_{x,h}(y) := \frac{1}{h} \phi\left(\frac{y - x}{h}\right).$$

The function  $\phi(y)$  decreases as  $1/y^2$  at infinity, hence it is in  $L^1(\mathbb{R}, dx)$  and so are  $\phi_{x,h}(y)$  with a uniform bound on the norm. However, these functions are not bounded, which makes it difficult to use them in our estimates. To finish the proof we will first obtain a bounded version of  $\phi_{x,h}(y)$  through the following averaging procedure.

Let  $\omega = g(x) dx$ . Choose  $x$  to be a Lebesgue point of  $g$  and a density point of  $E$ . Fixing sufficiently small  $h > 0$  we can find the set  $A(x, h) \subset (x - h, x - h/2) \cup (x + h/2, x + h)$  such that

- $A(x, h)$  consists of density points of  $E$ ,
- $|A(x, h)| \geq h/2$ ,
- $A(x, h)$  is symmetric with respect to  $x$ .

Let  $T_{x,h} := T := \{t \in (0, h) : x + t \in A(x, h)\}$ . Then  $|T| \geq h/4$ . Now put

$$\psi_{x,h}(y) := \frac{1}{|T|} \int_T \phi_{x,t}(y) dt.$$

By (22) one can see immediately that

$$(23) \quad |\psi_{x,h}| \leq \frac{M}{h} \text{ for some } M > 0 \text{ and } |\psi_{x,h}(y)| \leq C \frac{h}{y^2}, \text{ for } |y| > h.$$

Also, since

$$\int \phi dy = 0,$$

we have that

$$\int \psi_{x,h} dy = 0.$$

Therefore,

$$\begin{aligned} \left| \int g(y) \psi_{x,h}(y) dy \right| &= \left| \int (g(y) - g(x)) \psi_{x,h}(y) dy \right| \\ &\leq \int |g(y) - g(x)| |\psi_{x,h}(y)| dy. \end{aligned}$$

Now notice that (23) implies that  $|\psi_{x,h}|$  is majorized by an approximate unity (for instance, by a constant multiple of the Poisson kernel corresponding to  $z = x + ih$ ). Since  $x$  is a Lebesgue point for  $g(x)$ , this means that the last integral tends to 0 as  $h \rightarrow 0$ .

Looking at the definitions of  $T_{x,h}$  and  $\psi_{x,h}(y)$  we can see that

$$\int_{\mathbb{R}} g(y) \psi_{x,h}(y) dy = \frac{1}{|T_{x,h}|} \int_{T_{x,h}} \left[ \frac{1}{2t} (F(x+t) - F(x-t)) - \Re \int \frac{g(y)y}{x-it-y} \right],$$

where  $F(x)$  is the Green's function. As we mentioned before,  $F$  is zero at the density points of  $E$ . We conclude that

$$\Re \frac{1}{|T_{x,h}|} \int_{T_{x,h}} dt \int \frac{g(y) dy}{x-it-y} \rightarrow 0, \quad h \rightarrow 0+,$$

for a.e.  $x$  on the Borel support of  $g$ . Since the Cauchy integral of  $g$  has a limit a.e. we obtain that

$$\Re \int \frac{g(y) dy}{x-ih-y} \rightarrow 0, \quad h \rightarrow 0+. \quad \square$$

*Remark.* All reflectionless measures on  $\mathbb{R}$  discussed in this section, including those provided by Theorem 5.4 are absolutely continuous with respect to Lebesgue measure. One may wonder if there exist singular reflectionless measures. The answer is negative. More generally, as follows from a theorem from [8], if principal values of the Hilbert transform exist  $\mu$ -a.e. for a continuous  $\mu \in M(\mathbb{R})$  then  $\mu \ll dx$ .

### References

- [1] A. B. ALEKSANDROV, J. M. ANDERSON, AND A. NICOLAU, Inner functions, Bloch spaces and symmetric measures, *Proc. London Math. Soc. (3)* **79**(2) (1999), 318–352.
- [2] E. D. BELOKOLOS, A. I. BOBENKO, V. B. MATVEEV, AND V. Z. ÉNOL'SKIĬ, Algebro-geometric principles of superposition of finite-zone solutions of integrable nonlinear equations, (Russian), *Uspekhi Mat. Nauk* **41** (1986), no. 2(248), 3–42.
- [3] P. L. DUREN, “*Univalent functions*”, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, 1983.
- [4] L. C. EVANS AND R. F. GARIEPY, “*Measure theory and fine properties of functions*”, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
- [5] T. W. GAMELIN, “*Uniform algebras*”, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.
- [6] G. M. GOLUSIN, “*Geometrische Funktionentheorie*”, Hochschulbücher für Mathematik, Bd. **31**, VEB Deutscher Verlag der Wissenschaften, Berlin, 1957.
- [7] W. K. HAYMAN AND P. B. KENNEDY, “*Subharmonic functions*”, Vol. I, London Mathematical Society Monographs **9**, Academic



- Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976.
- [8] P. W. JONES AND A. G. POLTORATSKI, Asymptotic growth of Cauchy transforms, *Ann. Acad. Sci. Fenn. Math.* **29(1)** (2004), 99–120.
  - [9] M. G. KREĬN AND A. A. NUDEL'MAN, “*The Markov moment problem and extremal problems*”. Ideas and problems of P. L. Čebyšev and A. A. Markov and their further development, Translated from the Russian by D. Louvish, Translations of Mathematical Monographs **50**, American Mathematical Society, Providence, R.I., 1977.
  - [10] P. MATTILA, “*Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*”, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, Cambridge, 1995.
  - [11] M. S. MELNIKOV AND J. VERDERA, A geometric proof of the  $L^2$  boundedness of the Cauchy integral on Lipschitz graphs, *Internat. Math. Res. Notices* **7** (1995), 325–331.
  - [12] F. NAZAROV, S. TREIL, AND A. VOLBERG, Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces, *Internat. Math. Res. Notices* **15** (1997), 703–726.
  - [13] I. I. PRIVALOV, “*Graničnye svoïstva analitičeskikh funkcii*”, [Boundary properties of analytic functions], 2d ed., Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950.
  - [14] X. TOLSA,  $L^2$ -boundedness of the Cauchy integral operator for continuous measures, *Duke Math. J.* **98(2)** (1999), 269–304.
  - [15] X. TOLSA AND J. VERDERA, May the Cauchy transform of a non-trivial finite measure vanish on the support of the measure? *Ann. Acad. Sci. Fenn. Math.* **31(2)** (2006), 479–494.
  - [16] M. ZINSMEISTER, Espaces de Hardy et domaines de Denjoy, *Ark. Mat.* **27(2)** (1989), 363–378.

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